Differential Geometrical Formulation of Gauge Theory of Gravity

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Abstract

Differential geometric formulation of quantum gauge theory of gravity is studied in this paper. The quantum gauge theory of gravity which is proposed in the references hep-th/0109145 and hep-th/0112062 is formulated completely in the framework of traditional quantum field theory. In order to study the relationship between quantum gauge theory of gravity and traditional quantum gravity which is formulated in curved space, it is important to find the differential geometric formulation of quantum gauge theory of gravity. We first give out the correspondence between quantum gauge theory of gravity and differential geometry. Then we give out differential geometric formulation of quantum gauge theory of gravity.

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1 Introduction

It is known that, in classical Newton's theory of gravity, gravity is treated as physical interactions between two massive objects, and gravity does not affect the structure of space-time[1]. In Einstein's general theory of relativity, gravity is treated as geometry of space-time[2, 3]. In other words, in general relativity, gravity is not treated as a physical interactions, it is a part of structure of space-time. Inspired by Einstein's general theory of relativity, traditional relativistic theory of gravity and traditional canonical quantum theory of gravity is formulated in the framework of differential geometry[4, 5, 6, 7, 8, 9].

Recently, based on completely new notions and completely new methods, Wu proposes a new quantum gauge theory of gravity, which is based on gauge principle[10, 12]. The central idea is to use traditional gauge field theory to formulate quantum theory of gravity. This new quantum gauge theory of gravity is renormalizable[10, 11]. A strict formal proof on the renormalizability of the theory is also given in the reference [10, 11]. This new quantum gauge theory of gravity is formulated in the flat physical space-time, which is completely different from that of the traditional quantum gravity at first appearance. But this difference is not essential, for quantum gauge theory of gravity can also be formulated in curved space. In gauge theory of gravity, gravity is treated as physical interactions, but in this paper, gravity is treated as space-time geometry. It means that gravity has physics-geometry duality, which is the nature of gravitational interactions. In this paper, we first give out the correspondence between quantum gauge theory of gravity and differential geometry. Then, we give out the differential geometrical formulation of quantum gauge theory of gravity.

2 Correspondence Between Quantum Gauge Theory of Gravity and Differential Geometry

In gravitational gauge theory, there are two different kinds of space-time: one is the traditional Minkowski space-time which is the physical space-time, another is gravitational gauge group space-time which is not a physical space-time. Two space-times have different physical meanings. In differential geometry, there are also two different space-times. This can be seen from the tetrad field in Cartan form. In cartan form, there are two different space-times, one is base manifold, another is the tangent space in Cartan tetrad. The correspondence between these two different spaces in two different theories are: the traditional Minkowski space-time in gravitational gauge theory corresponds to the tangent space in Cartan tetrad; the gravitational gauge group space-time corresponds to the base manifold of differential geometry.

In gravitational gauge theory, the metric in Minkowski space-time is the flat metric $\eta^{\mu\nu}$, which corresponds to the flat metric η^{ab} in Cartan tetrad. The metric in gravitational gauge group space-time is the curved metric $g^{\alpha\beta}$ which corresponds to the curved metric $g^{\mu\nu}$ in differential geometry.

In gravitational gauge theory, G^{α}_{μ} is defined by

$$G^{\alpha}_{\mu} = \delta^{\alpha}_{\mu} - gC^{\alpha}_{\mu}, \tag{2.1}$$

which corresponds to the Cartan tetrad field $e_{a\cdot}^{\ \ \mu}$. $G^{-1\mu}_{\alpha}$ corresponds to reference frame field $e^{a}_{\cdot\mu}$. That is

$$G^{\alpha}_{\mu} \Longleftrightarrow e_{a\cdot}^{\ \ \mu}, \tag{2.2}$$

$$G^{-1\mu}_{\alpha} \Longleftrightarrow e^{a}_{\cdot\ \mu}. \tag{2.3}$$

Therefore, the following two relations have the same meaning,

$$g^{\alpha\beta} = \eta^{\mu\nu} G^{\alpha}_{\mu} G^{\beta}_{\nu} \Longleftrightarrow g^{\mu\nu} = \eta^{ab} e_{a\cdot}^{\ \ \mu} e_{b\cdot}^{\ \ \nu}, \tag{2.4}$$

where $g^{\alpha\beta}$ is the metric in gravitational gauge group space-time, and $g^{\mu\nu}$ is the metric of the base manifold in differential geometry. Similarly, the following two relations correspond to each other,

$$g_{\alpha\beta} = \eta_{\mu\nu} G^{-1\mu}_{\alpha} G^{-1\nu}_{\beta} \Longleftrightarrow g_{\mu\nu} = \eta_{ab} e^{a}_{\cdot\ \mu} e^{b}_{\cdot\ \nu}. \tag{2.5}$$

Define

$$\partial_a \stackrel{\triangle}{=} e_{a\cdot}^{\ \ \mu} \partial_\mu, \tag{2.6}$$

which is the derivative in Cartan tetrad. It corresponds to the gravitational gauge covariant derivative D_μ in gravitational gauge theory

$$D_\mu \Longleftrightarrow \partial_a. \tag{2.7}$$

In gravitational gauge theory, all matter field, such as scalar fields $\phi(x)$, Dirac field $\psi(x)$, vector field $A_\mu(x)$, $\cdots$, are fields in Minkowski space, which correspond to fields in Cartan tetrad. In other words, all matter fields in differential geometry are defined in Cartan tetrad.

Finally, what is the transformation in differential geometry which corresponds to the the gravitational gauge transformation? Under most general coordinate transformation, the transformation of Cartan tetrad field is :

$$e_{a.}^{\ \mu} \to e'^{\ \mu}_{a.} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} L_{a.}^{\ b} e_{b.}^{\ \nu}, \tag{2.8}$$

where $L_{a.}^{\ b}$ is the associated Lorentz transformation. As we have stated before, under gravitational gauge transformations, there is no associated Lorentz transformation. In other words, under gravitational gauge transformation,

$$L_{a.}^{\ b} = \delta_a^b. \tag{2.9}$$

Therefore, the gravitational gauge transformation of Cartan tetrad field is

$$e_{a.}^{\ \mu} \to e'^{\ \mu}_{a.} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} e_{a.}^{\ \nu}, \tag{2.10}$$

We call this transformation translation transformation. So, gravitational gauge transformation in gravitational gauge theory corresponds to the translation transformation in differential geometry. According to eq.(2.6), ∂_a does not change under translation transformation,

$$\partial_a \to \partial'_a = \partial_a. \tag{2.11}$$

Eq.(2.10) corresponds to the following transformation in gravitational gauge theory,

$$G^{\alpha}_{\mu} \to G'^{\alpha}_{\mu} = \Lambda^{\alpha}_{\ \beta}(\hat{U}_{\epsilon} G^{\beta}_{\mu}). \tag{2.12}$$

Eq.(2.11) corresponds to the following gravitational gauge transformation in gravitational gauge theory,

$$D_{\mu} \to D'_{\mu} = \hat{U}_{\epsilon} D_{\mu} \hat{U}_{\epsilon}^{-1}. \tag{2.13}$$

Finally, as a summary, we list some important correspondences between physical picture and geometry picture in the following table 1.

3 Differential Geometrical Formulation of Gravitational Gauge Theory

In differential geometrical formulation of gravitational gauge theory, all fields are expressed in Cartan orthogonal tetrad. In differential geometry, gravitational field

Quantum Gauge Theory of Gravity	Differential Geometry
gauge group space-time	base manifold
Minkowski space-time	tangent space in tetrad
index α	index μ
index μ	index a
G^{α}_{μ}	$e_{a.}^{\ \ \mu}$
$G^{-1\mu}_{\alpha}$	$e^{a}_{\cdot\ \mu}$
$g^{\alpha\beta} = \eta^{\mu\nu} G^{\alpha}_{\mu} G^{\beta}_{\nu}$	$g^{\mu\nu} = \eta^{ab} e_{a.}^{\ \ \mu} e_{b.}^{\ \ \nu}$
$g_{\alpha\beta} = \eta_{\mu\nu} G^{-1\mu}_{\alpha} G^{-1\nu}_{\beta}$	$g_{\mu\nu} = \eta_{ab} e^{a}_{\cdot\ \mu} e^{b}_{\cdot\ \nu}$
D_{μ}	∂_a
$F^{\alpha}_{\mu\nu}$	$\bar{\omega}^{\mu}_{ab}$
gravitational gauge transformation	translation transformation
gravitational gauge covariant	translation invariant
$\Lambda^{\alpha}_{\ \ \beta}$	$\frac{\partial x'^{\mu}}{\partial x^{\nu}}$
$\Lambda_{\alpha}^{\ \ \beta}$	$\frac{\partial x^{\nu}}{\partial x'^{\mu}}$
$G^{\alpha}_{\mu} \to G'^{\alpha}_{\mu} = \Lambda^{\alpha}_{\ \ \beta} (\hat{U}_{\epsilon} G^{\beta}_{\mu})$	$e_{a.}^{\ \ \mu} \to e'^{\ \ \mu}_{a.} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} e_{a.}^{\ \ \nu}$
$G^{-1\mu}_{\alpha} \to G'^{-1\mu}_{\alpha} = \Lambda_{\alpha}^{\ \ \beta} (\hat{U}_{\epsilon} G^{-1\mu}_{\beta})$	$e^{a}_{\cdot\ \mu} \to e'^{a}_{\cdot\ \mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} e^{a}_{\cdot\ \nu}$
$g^{\alpha\beta} \to g'^{\alpha\beta} = \Lambda^{\alpha}_{\ \ \alpha_1} \Lambda^{\beta}_{\ \ \beta_1} (\hat{U}_{\epsilon} g^{\alpha_1\beta_1})$	$g^{\mu\nu} \to g'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\mu_1}} \frac{\partial x'^{\nu}}{\partial x^{\nu_1}} g^{\mu_1\nu_1}$
$g_{\alpha\beta} \to g'_{\alpha\beta} = \Lambda_{\alpha}^{\ \ \alpha_1} \Lambda_{\beta}^{\ \ \beta_1} (\hat{U}_{\epsilon} g_{\alpha_1\beta_1})$	$g_{\mu\nu} \to g'_{\mu\nu} = \frac{\partial x^{\mu_1}}{\partial x'^{\mu}} \frac{\partial x^{\nu_1}}{\partial x'^{\nu}} g_{\mu_1\nu_1}$
$D_{\mu} \to D'_{\mu} = \hat{U}_{\epsilon} D_{\mu} \hat{U}^{-1}_{\epsilon}$	$\partial_a \to \partial'_a = \partial_a$
$F^{\alpha}_{\mu\nu} \to F'^{\alpha}_{\mu\nu} = \Lambda^{\alpha}_{\ \ \beta} (\hat{U}_{\epsilon} F^{\beta}_{\mu\nu})$	$\bar{\omega}^{\mu}_{ab} \to \bar{\omega}'^{\mu}_{ab} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \bar{\omega}^{\nu}_{ab}$

Table 1: Correspondence between two pictures of gravity.

is represented by tetrad field $e_{a\cdot}^{\ \mu}$. The field strength of gravitational field is denoted by $\bar{\omega}_{ab}^{\mu}$,

$$\bar{\omega}_{ab}^{\mu} = -\frac{1}{g}\left[(\partial_a e_{b\cdot}^{\ \mu}) - (\partial_b e_{a\cdot}^{\ \mu}) \right]. \tag{3.1}$$

$\bar{\omega}_{ab}^{\mu}$ corresponds to the field strength tensor $F_{\mu\nu}^{\alpha}$ in gravitational gauge theory. Under translation transformation, $\bar{\omega}_{ab}^{\mu}$ transforms as

$$\bar{\omega}_{ab}^{\mu} \to \bar{\omega}_{ab}^{\prime\mu} = \frac{\partial x^{\prime\mu}}{\partial x^{\nu}} \bar{\omega}_{ab}^{\nu}. \tag{3.2}$$

It is known that, in differential geometry, covariant derivative of cartan tetrad vanishes

$$D_\mu e_{\cdot\ \nu}^{a} = 0. \tag{3.3}$$

It gives out the following relation

$$\overset{\sim}{\Gamma}{}_{bc}^{a} = e_{\cdot\ \lambda}^{a} e_{b\cdot}^{\ \nu} e_{c\cdot}^{\ \mu} \Gamma_{\mu\nu}^{\lambda} + e_{\cdot\ \nu}^{a} (\partial_\mu e_{b\cdot}^{\ \nu}) e_{c\cdot}^{\ \mu}, \tag{3.4}$$

where $\Gamma_{\mu\nu}^{\lambda}$ is the affine connexion and $\overset{\sim}{\Gamma}{}_{bc}^{a}$ is the Cartan connexion. From this relation, we can obtain

$$\tilde{T}{}_{bc}^{a} = e_{\cdot\ \lambda}^{a} e_{b\ \cdot}^{\ \mu} e_{c\ \cdot}^{\ \nu} T_{\mu\nu}^{\lambda} + g e_{\cdot\ \mu}^{a} \bar{\omega}_{bc}^{\mu} \tag{3.5}$$

where $T_{\mu\nu}^{\lambda}$ and $\tilde{T}{}_{bc}^{a}$ are torsion tensors

$$\tilde{T}{}_{bc}^{a} = \overset{\sim}{\Gamma}{}_{bc}^{a} - \overset{\sim}{\Gamma}{}_{cb}^{a}, \tag{3.6}$$

$$T_{\mu\nu}^{\lambda} = \Gamma_{\mu\nu}^{\lambda} - \Gamma_{\nu\mu}^{\lambda}. \tag{3.7}$$

For gravitational gauge theory, the affine connexion is the Christoffel connexion. For Christoffel connexion, the torsion $T_{\mu\nu}^{\lambda}$ vanish. Then

$$\tilde{T}{}_{bc}^{a} = g e_{\cdot\ \mu}^{a} \bar{\omega}_{bc}^{\mu}. \tag{3.8}$$

It seems that the field strength of gravitational field is related to the torsion of Cartan connexion.

Translation transformation of metric tensor $g_{\mu\nu}$ is

$$g_{\mu\nu} \to g'_{\mu\nu} = \frac{\partial x^{\mu_1}}{\partial x^{\prime\mu}} \frac{\partial x^{\nu_1}}{\partial x^{\prime\nu}} g_{\mu_1\nu_1}. \tag{3.9}$$

The metric in Cartan tetrad is denoted as η^{ab}, which is the traditional Minkowski metric. Under translation transformation, η^{ab} is invariant

$$\eta^{ab} \to \eta^{\prime ab} = \eta^{ab}. \tag{3.10}$$

The lagrangian $\mathcal{L}$ for pure gravitational field is selected as

$$\mathcal{L} = -\frac{1}{4}\eta^{ac}\eta^{bd}g_{\mu\nu}\bar{\omega}^{\mu}_{ab}\bar{\omega}^{\nu}_{cd}. \tag{3.11}$$

Using eq.(2.5) and eq.(3.8), we can change the above lagrangian density into the following form

$$\mathcal{L} = -\frac{1}{4g^2}\eta^{ac}\eta^{bd}\eta_{ef}\ \tilde{T}{}^{e}_{ab}\tilde{T}{}^{f}_{cd}\ . \tag{3.12}$$

It is easy to prove that this lagrangian is invariant under translation transformation. The concept of translation invariant in differential geometry corresponds to the concept of gravitational gauge covariant in gravitational gauge theory.

In order to introduce translation invariant action of the system, we need to introduce a factor which is denoted as $J(C)$ in gravitational gauge theory. In this paper, $J(C)$ is selected as

$$J(C) = \sqrt{-\det(g_{\mu\nu})}. \tag{3.13}$$

The action of the system is defined by

$$S = \int \mathrm{d}^4x\ J(C)\ \mathcal{L}. \tag{3.14}$$

This action is invariant under translation transformation.

For real scalar field ϕ, its gravitational interactions are described by

$$\mathcal{L} = -\frac{1}{2}(\partial_a\phi)(\partial_b\phi) - \frac{1}{2}m^2\phi^2, \tag{3.15}$$

where m is the mass of scalar. For complex scalar field, its lagrangian is selected to be

$$\mathcal{L} = -(\partial_a\phi)^*(\partial_b\phi) - m^2\phi^*\phi. \tag{3.16}$$

Under translation transformations, $\phi(x)$ and $\partial_a\phi(x)$ transform as

$$\phi(x) \to \phi'(x') = \phi(x), \tag{3.17}$$

$$\partial_a\phi(x) \to \partial'_a\phi'(x') = \partial_a\phi(x). \tag{3.18}$$

Using eq.(3.17) and eq.(3.18), we can prove that the lagrangians which are given by eq.(3.15) and eq.(3.16) are translation invariant.

For Dirac field, its lagrangian is given by

$$\mathcal{L} = -\bar{\psi}(\gamma^a \partial_a + m)\psi. \tag{3.19}$$

Under translation transformations,

$$\psi(x) \to \psi'(x') = \psi(x), \tag{3.20}$$

$$\partial_a \psi(x) \to \partial'_a \psi'(x') = \partial_a \psi(x), \tag{3.21}$$

$$\gamma^a \to \gamma'^a = \gamma^a. \tag{3.22}$$

Under these transformations, the lagrangian given by eq.(3.19) is invariant.

For vector field, the lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}\eta^{ac}\eta^{bd} A_{ab} A_{cd} - \frac{m^2}{2}\eta^{ab} A_a A_b, \tag{3.23}$$

where

$$A_{ab} = \partial_a A_b - \partial_b A_a, \tag{3.24}$$

which is the field strength of vector field. This lagrangian is invariant under the following translation transformations

$$A_a(x) \to A'_a(x') = A_a(x), \tag{3.25}$$

$$A_{ab}(x) \to A'_{ab}(x') = A_{ab}(x), \tag{3.26}$$

$$\eta^{ab} \to \eta'^{ab} = \eta^{ab}. \tag{3.27}$$

For $U(1)$ gauge field, its lagrangian is[10, 11, 13]

$$\mathcal{L} = -\bar{\psi}\left[\gamma^a(\partial_a - ieA_a) + m\right]\psi - \frac{1}{4}\eta^{ac}\eta^{bd}\mathbf{A}_{ab}\mathbf{A}_{cd}, \tag{3.28}$$

where

$$\mathbf{A}_{ab} = A_{ab} + g e^{c}_{\cdot\ \mu} A_c \bar{\omega}^{\mu}_{ab}, \tag{3.29}$$

$$A_{ab} = \partial_a A_b - \partial_b A_a, \tag{3.30}$$

In eq.(3.28), e is the coupling constant for $U(1)$ gauge interactions. This lagrangian is invariant under the following translation transformation,

$$A_a(x) \to A'_a(x') = A_a(x), \tag{3.31}$$

$$\mathbf{A}_{ab}(x) \to \mathbf{A}'_{ab}(x') = \mathbf{A}_{ab}(x). \tag{3.32}$$

It is also invariant under local $U(1)$ gauge transformation[10, 11, 13].

For $SU(N)$ non-Abel gaueg field, its lagrangian is[10, 11, 13]

$$\mathcal{L} = -\bar{\psi}\left[\gamma^a(\partial_a - ig_c A_a) + m\right]\psi - \frac{1}{4}\eta^{ac}\eta^{bd}\mathbf{A}^i_{ab}\mathbf{A}^i_{cd}, \tag{3.33}$$

where

$$\mathbf{A}^i_{ab} = A^i_{ab} + g e^c_{\cdot\ \mu} A^i_c \bar{\omega}^\mu_{ab}, \tag{3.34}$$

$$A^i_{ab} = \partial_a A_b - \partial_b A_a + g_c f_{ijk} A^j_a A^k_b. \tag{3.35}$$

In above relations, g_c is the coupling constant for $SU(N)$ non-Able gauge interactions. It can be proved the this lagrangian is invariant under $SU(N)$ gauge transformation and translation transformation[10, 11, 12, 13].

4 Summary

In this paper, the geometrical formulation of gauge theory of gravity is studied, which is performed in the geometrical formulation of gravity[10, 12]. In this picture, we can see that gravitational field is put into the structure of space-time and there is no physical gravitational interactions in space-time.

In gravitational gauge theory, all matter field, such as scalar fields $\phi(x)$, Dirac field $\psi(x)$, vector field $A_\mu(x)$, $\cdots$, are fields in Minkowski space, which correspond to fields in Cartan tetrad. In other words, all matter fields in differential geometry are defined in Cartan tetrad.

In gravitational gauge theory, the symmetry transformation is gravitational gauge transformation, while in differential geometry, the corresponding symmetry transformation is translation transformation. The concept of translation invariant in differential geometry corresponds to the concept of gravitational gauge covariant in gravitational gauge theory.

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